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Discrete Mathematics: A Great Curriculum Connector

IMSA Professional Learning Day

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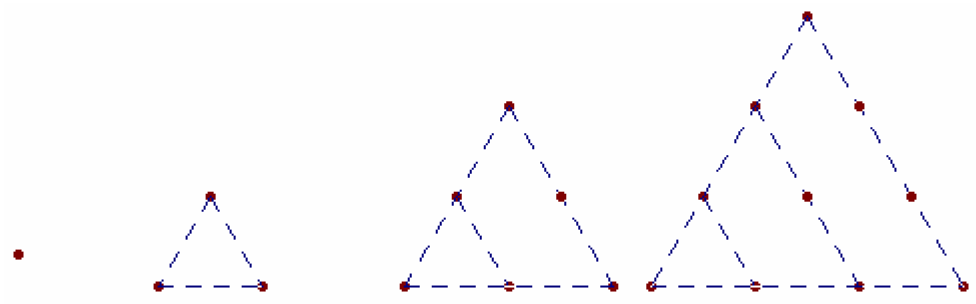
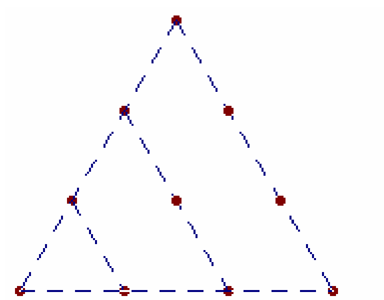
Below are drawings of the first four Triangular Numbers. (Can you guess why they are called Triangular Numbers?)



In the space below, use the fourth Triangular Number to help you draw the fifth Triangular Number.



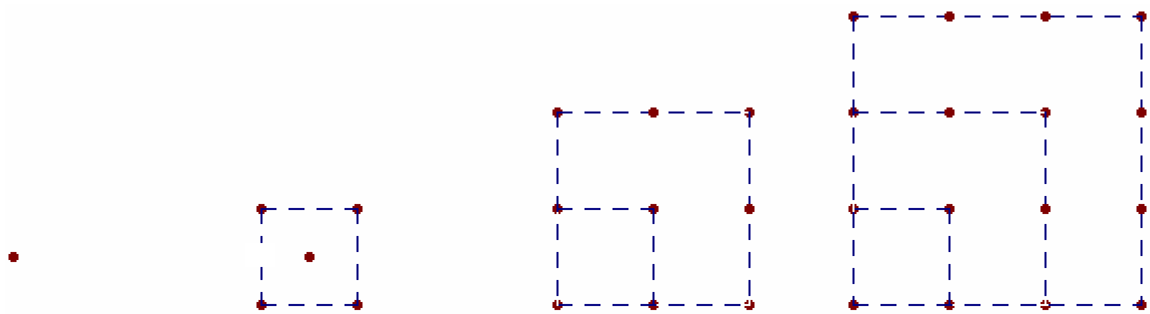
Now, see if you can complete the table below.

Triangular Numbers									
	1st	2nd	3rd	4th	5th	6th	7th	…	*n*th
Number of dots	1	3	6	10				…	

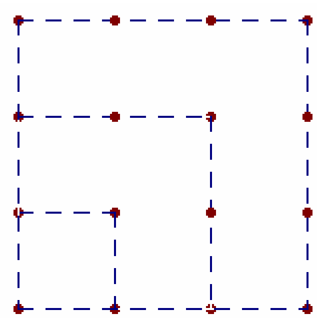
If you haven't figured out a method for determining the nth Triangular Number, consider using the process of finite differences to help you discover it.

Polygonal Numbers Worksheet Name: ______________________________

Below are drawings of the first four Square Numbers. (Can you guess why they are called Square Numbers?)



In the space below, use the fourth Square Number to help you draw the fifth Square Number.



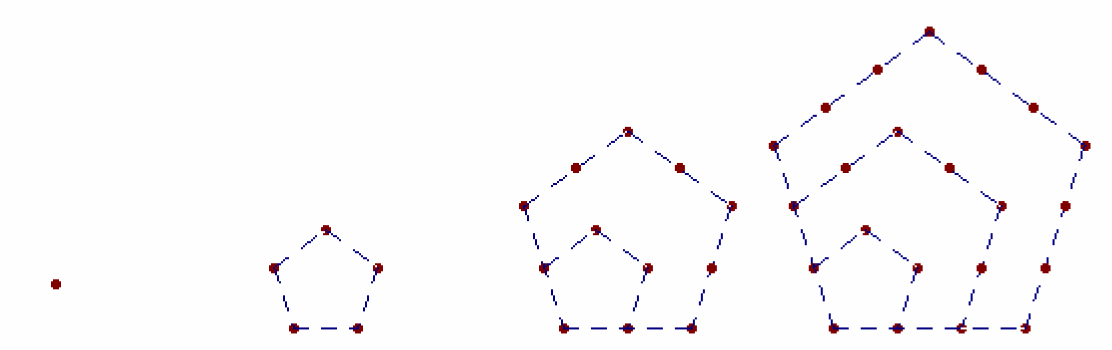
Now, see if you can complete the table below.

Square Numbers									
	1st	2nd	3rd	4th	5th	6th	7th	…	*n*th
Number of dots	1	4	9	16				…	

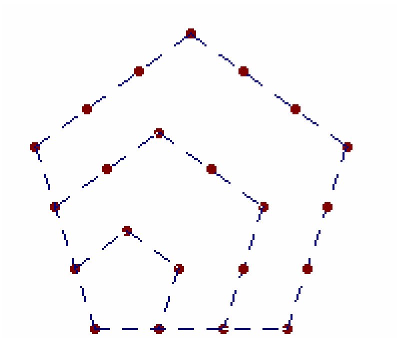
If you haven't figured out a method for determining the *n*th Square Number, consider using the process of finite differences to help you discover it.

Polygonal Numbers Worksheet Name: ______________________________

Below are drawings of the first four Pentagonal Numbers. (Can you guess why they are called Pentagonal Numbers?)



In the space below, use the fourth Pentagonal Number to help you draw the fifth Pentagonal Number.



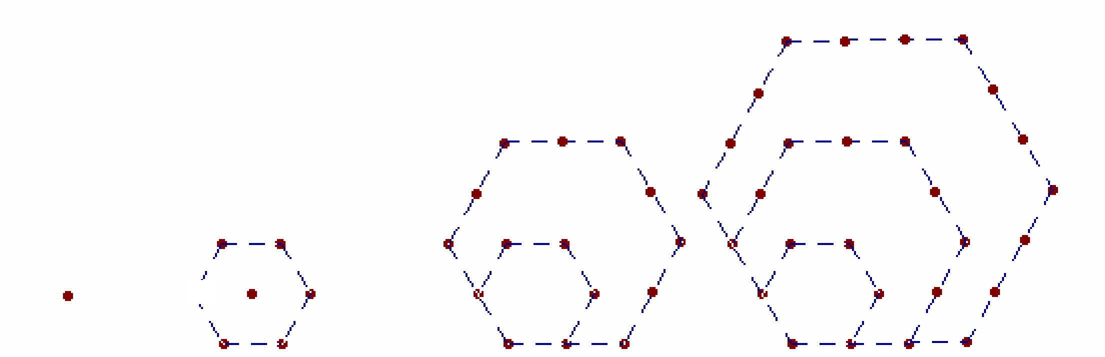
Now, see if you can complete the table below.

Pentagonal Numbers									
	1st	2nd	3rd	4th	5th	6th	7th	…	*n*th
Number of dots	1	5	12	22				…	

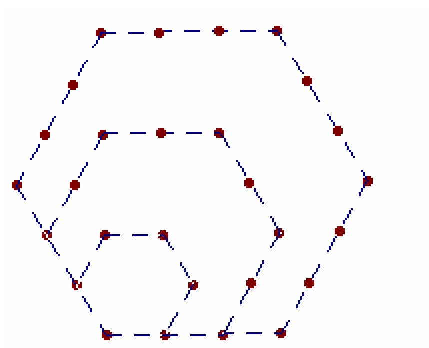
If you haven't figured out a method for determining the *n*th Pentagonal Number, consider using the process of finite differences to help you discover it.

Polygonal Numbers Worksheet Name: ________________________________

Below are drawings of the first four Hexagonal Numbers. (Can you guess why they are called Hexagonal Numbers?)



In the space below, use the fourth Hexagonal Number to help you draw the fifth Hexagonal Number.



Now, see if you can complete the table below.

Hexagonal Numbers									
	1st	2nd	3rd	4th	5th	6th	7th	…	*n*th
Number of dots	1	6	15	28				…	

If you haven't figured out a method for determining the *n*th Hexagonal Number, consider using the process of finite differences to help you discover it.

Polygonal Numbers Worksheet Name: ____________________

1	**2**	**3**	**4**	**5**	**6**	**7**	**…**	***n***
1	1						…	
1	2						…	
1	3	6	10	15	21	28	…	$\frac{n(n+1)}{2}$
1	4	9	16	25	36	49	…	n^2
1	5	12	22	35	51	70	…	$\frac{n(3n-1)}{2}$
1	6	15	28	45	66	91	…	$n(2n-1)$
1	7						…	
1	8	21	40	65	96	133	…	$n(3n-2)$
1	9							
1	10							
1	12							

1. In the table above, fill in the third through seventh polygonal numbers for the:
   a. heptagonal numbers (7th row) b. nonagonal numbers (9th row)
   c. decagonal numbers (10th row) d. dodecagonal numbers (12th row)
   e. "two"-gonal (2nd row) e. "one"-gonal numbers (1st row)

2. Find each general polygonal number and enter it in the appropriate position in the table above. Recall that P(m, n) refers to the nth "m"-gonal number so that P(3, n) = $\frac{n(n+1)}{2}$, and P(4, n) = n^2.
   a. P(7, n) b. P(9, n) c. P(10, n) d. P(12, n) e. P(2, n) f. P(1, n)

3. Use the formulas for the *n*th "*m*"-gonal numbers in the table above to derive a formula for P(*m*, *n*), the general *n*th "*m*"-gonal number.

4. Look for (at least) one pattern that relates entries from different rows of polygonal number (like the relationship between triangular and square numbers we discussed). In the space below describe this relationship (and any others you find).

The Josephus Problem

Flavius Josephus was a Jewish historian during the Roman-Jewish war of the first century. Through his writings comes the following macabre story:

The Romans had chased a group of 41 Jews into a cave and were about to attack. Rather than dying at the hands of their enemy, the group chose to commit suicide one by one. Legend has it though, that they decided to go around their circle of 41 individuals and eliminate every other person until no one was left. Who was the last to die?

Suppose n people numbered 1 to n stand around in a circle. Starting from 1, every second remaining person is seated (we can't have them killed, can we) until only one is left standing.

Who is the only one left standing?

Call this person $J(n)$.

Determine $J(n)$ for n = 1 to 32

n	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
***J*(*n*)**								
n	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**
***J*(*n*)**								

n	**17**	**18**	**19**	**20**	**21**	**22**	**23**	**24**
***J*(*n*)**								
n	**25**	**26**	**27**	**28**	**29**	**30**	**31**	**32**
***J*(*n*)**								

Hopefully, you've noticed some patterns in these numbers. For example, it should be clear that $J(n)$ must always be odd. It also appears that $J(n) = 1$ if n is a power of 2. Let's see if we can use these patterns, plus some others to generalize our results to determine $J(n)$ for any natural number, n.

Pouring Problems

1. Two men wandering in the desert come upon a trail. In hope of finding help sooner, they decide to split up, each taking half of their remaining water, and set off in opposite directions along the trail. They have only one 12-cup canteen full of water and two empty canteens that hold 9 cups and 5 cups respectively. The only way for them to measure the water is by pouring water from one canteen to another until the first canteen is empty or the second canteen is full (there are no volume markings on the canteens). Is it possible for the two men to measure exactly 6 cups of water into each of the two larger canteens? If so, show how they can do it by listing the pours between the three canteens in the table below. If not, explain why not.

Cups of water in 12-cup canteen	Cups of water in 9-cup canteen	Cups of water in 5-cup canteen
12	0	0

2. In the process of pouring water between the canteens to get 6 cups of water into each of the two larger canteens, you generated other volumes along the way. Did you come up with all the different possible volumes between 0 cups and 12 cups? If not, continue the process (in the table above) until you have.

3. Suppose instead that our two lost men only have a 9-cup canteen and a 5-cup canteen, and that they happen upon an oasis when they reach the trail and decide to split up. Now, the men can fill either canteen with water from the oasis, pour all the water out of a canteen into the oasis, or pour water from one canteen to another until the first canteen is empty or the second canteen is full. Is it possible for the two men to measure all the different possible volumes between 0 cups and 9 cups? If so, show how they can do it by listing the pours between the two canteens in the table below. If not, explain why not. Assume they start with both canteens empty.

Cups of water in 9-cup canteen	Cups of water in 5-cup canteen
0	0

4. Would your answer to the question in problem #3 be different if the two men only have a 9-cup canteen and a 6-cup canteen? Explain why or why not.

Here are some additional questions to consider:

5. Given an unlimited supply of water and two empty unmarked canteens of different volumes:
   a. When pouring back and forth be between the canteens and the unlimited supply of water, will it always be possible to obtain every volume up to the volume of the largest canteen?

   b. If it is not always possible, then under what conditions will it be possible?

   c. If it is not always possible, then what volumes can be obtained when not all can be obtained?

6. Is there some way to use *geometry* to determine all the volumes that can be obtained when pour between two unmarked canteens of different volumes and an unlimited water problem? How about between three unmarked canteens of different volumes when the canteen of largest volume is full and the other two canteens are empty? (Of course, there is, otherwise I wouldn't have asked the question! And trust me, it is a really neat connection.)

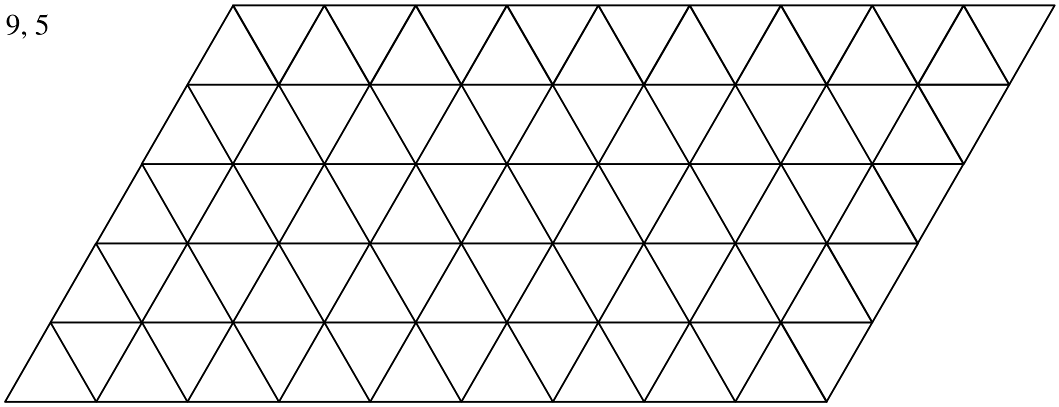
7. Given three unmarked canteens of different volumes with the canteen of largest volume full and the other two canteens empty:
   a. When pouring back and forth be between the canteens, will it always be possible to obtain every volume up to the volume of the largest canteen?

   b. If it is not always possible, then under what conditions will it be possible?

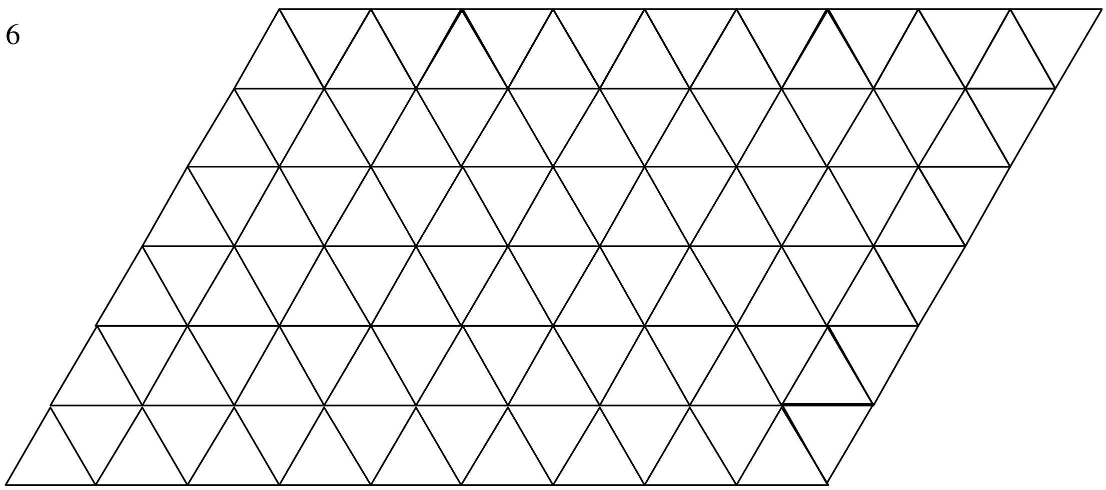
   c. If it is not always possible, then what volumes can be obtained when not all can be obtained?

Pouring Problems Worksheet Name: ______________________________

9, 5



9, 6



12, 9, 5